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Aggregation of a Two-Sector Economy with Heterogeneous Capitals *

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Preliminary Version

Abstract

This study shows the possibility of aggregating an economy that has two production sectors producing two heterogeneous (or sector specific) goods capitals with two heterogeneous (or sector specific) capitals to the economy that has one sector producing one good, which is so-called a macro economy. The aggregation process requires two kinds of consistency. The first consistency is the one between the factor demands in the economy before and after the aggregation. The second consistency is the total value of the goods in equilibria of economies before and after the aggregation. We show that the aggregation of a economy is possible with satisfying these consistencies.

Keywords: aggregation, heterogeneous capital, two-sector economy

JEL Classification: B41, D24, E23, O41

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1 Introduction

A macro production function plays great roles in capturing economic performances of an industry, a country or anything that individual economic agents are gathered to one agent. The assumption behind the macro production function is that there are individual sectoral or firm-level production functions (micro production functions), which are aggregated consistently into a macro production function. However, this assumption is not always clearly justified.

There exists a body of research known as the Cambridge Controversy, which has accumulated studies on the aggregation of production technology. From early works such as May (1946) to Fisher (1993) and more recent studies like Felipe and Fischer (2003) and Felipe and McCombie (2013), these studies have generally yielded negative results regarding aggregation.

However, Baquee and Farhi (2019) demonstrated the possibility of an aggregate production functions from a new perspective. Their discussion follows the steps of

1. Assuming the existence of an "aggregator" that is a function aggregating a vector of total net supplies to a real value,

2. Supposing a aggregator maximizing problem under resource constraints,

3. Solving the problem and describing the aggregator by the amount of initially endowed resources (capital and labor) by using the solution, i,e., obtaining a composite function.

4. Calling the composite function as an aggregate production function.

The major difference between the discussion of Cambridge controversy and that of Baquee and Farhi (2019) lies in how they capture an aggregate production function. The former captures an aggregate production function solely by utilizing individual producers' information, meaning their production technologies and their behaviors. On the other hand, the latter does it by the information of both producers' and information of aggregator function. The discussion of Baquee and Farhi (2019) is significant in the sense that they provide a possible way to construct an aggregate production function.

However, Baquee and Farhi (2019) neither identify the properties of functions that can be called a aggregator, nor preserve the consistency between the individual production functions and the aggregate production function.

Eventually, they do not discuss the aggregation of individual economies to an aggregate economy.

The consistency needs to be checked from the two aspects: the aggregated commodity and the aggregated commodity price. We can consider two types of economies: an economy with the aggregate producer, and an economy with individual producers. In the former economy, the aggregate producer maximizes its profit and determine the amount of the aggregated commodity supplied and the amount of production factors demanded given the prices the aggregated commodity and production factors. In the latter economy, the amount of individual commodities supplied and the total amount of factors demanded are determined in the same fashion. It is meaningful to examine the consistencies between these economies. It is also meaningful to examine the relationship between the price of aggregated commodity and that of the individual commodities.

Doi, Fujii, Horie, Iritani, Sato, and Yasuoka (2021) show that there exists an aggregate production function in a two-sector economy with homogeneous capitals in a fashion preserving the two types of consistencies by way of Cobb-Douglas example. This study extends Doi, Fujii, Horie, Iritani, Sato, and Yasuoka(2021) by introducing heterogeneous capitals to a more general setting of the function with the homogeneity of degree one. This extension requires us to conduct the aggregation of heterogeneous capitals to an aggregated capital in addition to the aggregation of a two-sector economy to a one-sector economy as is done by Doi, Fujii, Horie, Iritani, Sato, and Yasuoka (2021). We conduct these two types of aggregation and show the existence of an aggregate production function.

2 Two-Sector Model with Heterogeneous Capitals

2.1 Production Technologies

Let us consider an economy with two production sectors. We assume that each sector produces a sector specific good. The production technology of the i-th sector is described by the production function

$$Y_i = F_i(K_i, L_i),\ (K_i, L_i) \in \mathbb{R}^2_+,\ i = 1, 2,$$

where Y_i, K_i, and L_i denote the amounts of the products, the capital and the labor of the i-th sector respectively. Sector 1 and 2 utilize the sector-specific capitals, namely the capitals are heterogeneous, and homogeneous labor.

Assumption 1 *We assume each production function F_i satisfies following conditions* A1, A2, A3, *and* A4, $i = 1, 2$.

A1 *Each production function F_i is twice continuously differentiable, concave and homogenous of degree one.*

By homogeneity of F_i, we can represent the production function F_i by $K_i g_i(\ell_i)$, where $g_i(\ell_i) = F_i(1, \ell_i)$ and $\ell_i = L_i/K_i$.

A2 *For any positive K_i, L_i, it holds that $F_i(0, L_i) = F_i(K_i, 0) = 0$, $i = 1, 2$.*

A3 *F_i is twice continuously differentiable with respect to $(K_i, L_i) \in \mathbb{R}^2_{++}$ and satisfies:*

$$\frac{\partial F_i}{\partial L_i} > 0,\ \frac{\partial F_i}{\partial K_i} > 0,\ \frac{\partial^2 F_i}{\partial {L_i}^2} < 0,\ \frac{\partial^2 F_i}{\partial {K_i}^2} < 0,\ i = 1, 2.$$

A4 *F_i is well-behaved, that is, has following properties:*

$$\lim_{\ell_i \to 0} {g_i}'(\ell_i) = \infty \text{ and } \lim_{\ell_i \to \infty} {g_i}'(\ell_i) = 0,\ i = 1, 2.$$

2.2 Equilibrium of a Two-Sector Economy

Let K_i^d, and $\bar{K}_i$ denote sector i's demand for the sector-specific capital, and the amount of initial endowment of the capital respectively. The assumption implies the equilibrium condition of the capital market is

$$K_i^d = \bar{K}_i, \quad i = 1, 2.$$

We assume that the sector-specific capital market is competitive, and thus the values of the marginal productivity and the rental ratios of the sector-specific capital are respectively equal in the equilibrium. Hereafter, we use K_i (> 0) for the notation of amount of the sector i-specific capital initially endowed to the economy instead of $\bar{K}_i$. The production function is given by (**??**). The amount of labor initially endowed to the economy is $L(> 0)$. Suppose that the production functions F_1, F_2 satisfy Assumption 1.

We begin with describing an economy with two goods, two sectors, two capitals, and one labor. We call this economy as an $\mathcal{E}_2$-economy. Let (α_1, α_2) denote the vector of given expenditure coefficients ($\alpha_i > 0$, $i = 1, 2$). By this coefficients, we assume that there exists one consumer that expends its α_i of income to its demand for good i, where $\alpha_1 + \alpha_2 = 1$. Now we can say that $((K_1, K_2, L), (F_i)_{i=1}^2, (\alpha_i)_{i=1}^2)$ economy is an $\mathcal{E}_2$-economy

Now we are ready to discuss the general equilibrium of $\mathcal{E}_2 = ((K_1, K_2, L), (F_i)_{i=1}^2, (\alpha_i)_{i=1}^2)$.

Definition 1 *Suppose an $\mathcal{E}_2$-economy, $((K_1, K_2, L), (F_i)_{i=1}^2, (\alpha_i)_{i=1}^2)$ is given. A pair consisting of a pair of price vector and allocation, $((\rho_i^*, p_i^*)_{i=1}^2, (Y_i^*, K_i^*, L_i^*)_{i=1}^2)$ is the general equilibrium in production of $\mathcal{E}_2$-economy if and only if the pair satisfies following conditions*(i), (ii), *and* (iii).

(i) (Y_i^*, K_i^*, L_i^*) *is a solution to the problem*[1],

$$\max_{Y_i, \tilde{K}_i, L_i} p_i^* Y_i - \rho_i^* \tilde{K}_i - L_i \ \text{ subject to } Y_i = F_i(\tilde{K}_i, L_i), \ i = 1, 2 \tag{1}$$

(ii) *Factor markets are in balance.*

$$L_1^* + L_2^* = L, \quad K_i^* = K_i, \ i = 1, 2 \tag{2}$$

(iii) *Commodtity markets are in balance.*

$$\alpha_i(\rho_1^* K_1 + \rho_2^* K_2 + L) = p_i^* F_i(K_i^*, L_i^*), \ i = 1, 2 \tag{3}$$

In definition 1 では, prices are measured by the wage level, meaning that ρ_i stands for the rental-wage ratio of the$i$-th sector specific capital, and p_i stands for the price-wage ratio of the i-th commodity . We assume that the proportion α_i of the real total income $\rho_1^* K_1 + \rho_2^* K_2 + L$ is spent for the purchase of the i-th commodity. Namely, the LHS of (3) means the monetary value of the quantity demanded for the i-th commodity, and the RHS of (3)is the monetary value of the quantity supplied of the i-th commodity. As you can see, we limit our interest on the production side by simplyfying the demand side of the commodity market, and we call this general equilibrium defined by 1 as “the general equilibrium in production”.

Next, we see the factor demand functions that are necessary for the discussion of the aggregation, and we prove the existence of the general equilibrium in production after that.

[1]We let $\tilde{K}_i$ denote the amount of capital instead of K_i, because we let K_i denote the amount of initial endowment of capital.

2.3 Demand, Supply, and Walrus Law

Let the price-wage ratio and the rental-wage ratio be $p_1, p_2, \rho_1,\ \rho_2$ respectively. The i-th sector's profit maximization behavior leads the following marginal condition of

$$\begin{aligned} 1\ &= p_i \frac{\partial F_i}{\partial \tilde{L}_i}(\tilde{K}_i, \tilde{L}_i) \\ \rho_i &= p_i \frac{\partial F_i}{\partial \tilde{K}_i}(\tilde{K}_i, \tilde{L}_i),\ i = 1, 2. \end{aligned}$$

From the homogeneity of degree one of F_i, $F_i(\tilde{K}_i, \tilde{L}_i) = \tilde{K}_i g_i(\tilde{L}_i/\tilde{K}_i)$, and if it is given as $\ell_i = \tilde{L}_i/\tilde{K}_i$, we have

$$\begin{aligned} &\frac{\partial F_i}{\partial \tilde{L}_i}(\tilde{K}_i, \tilde{L}_i) = g_i'(\ell_i), \\ &\frac{\partial F_i}{\partial \tilde{K}_i}(\tilde{K}_i, \tilde{L}_i) = g_i(\ell_i) - \ell_i g_i'(\ell_i), \\ &\frac{\partial^2 F_i}{\partial \tilde{L}_i^2} = \frac{g_i''(\ell_i)}{\tilde{K}_i} < 0, \end{aligned}$$

and we have the marginal condition

$$1 = p_i g_i'(\ell_i),\ \rho_i = p_i\{g_i(\ell_i) - \ell_i g_i'(\ell_i)\},$$

and, thus, we obtain

$$\rho_i = \frac{g_i(\ell_i) - \ell_i g_i'(\ell_i)}{g_i'(\ell_i)} = \frac{g_i(\ell_i)}{g_i'(\ell_i)} - \ell_i. \tag{4}$$

The RHS of (4) is a function of ℓ_i. Because g_i is a strictly increasing concave function, the numerator of the RHS of (4) is increasing and the denominator of the RHS of (4) is decreasing in ℓ_i. That is to say, from (A4),, the RHS of (4) is increasing in ℓ_i, converges to zero as ℓ_i goes to zero, and diverges to infinity as ℓ_i goes to infinity, Therefore, (4) has a unique solution of $\ell_i(\rho_i)$. From the inverse function theorem, $\ell_i(\rho_i)$ is differentiable by ρ_i. Now it can be said that

$$1 = \ell_i' - \frac{g_i g_i'' \ell_i'}{(g_i')^2} - \ell_i' \ \Rightarrow\ \ell_i' = -\frac{(g_i')^2}{g_i g_i''} > 0,$$

and

$$\frac{1}{\rho_i + \ell_i(\rho_i)} = \frac{g_i'(\ell_i(\rho_i))}{g_i(\ell_i(\rho_i))}. \tag{5}$$

Moreover, the commodity price can be described as a function of ρ_i,

$$p_i(\rho_i) = \frac{1}{g_i'(\ell_i(\rho_i))} = \frac{1}{\frac{\partial F}{\partial L_i}(\tilde{K}_i, \ell_i(\rho_i)\tilde{K}_i)},\ \forall \tilde{K}_i > 0, \tag{6}$$

and note that

$$\tilde{K}_i \times p_i(\rho_i) g_i(\ell_i(\rho_i)) = p_i(\rho_i) F_i(\tilde{K}_i, \tilde{K}_i \ell_i(\rho_i)) = \frac{F_i(\tilde{K}_i, \tilde{K}_i \ell_i(\rho_i))}{\frac{\partial F}{\partial L_i}(\tilde{K}_i, \ell_i(\rho_i)\tilde{K}_i)} \tag{7}$$

$$p_i(\rho_i) g_i(\ell_i(\rho_i)) = \frac{g_i(\ell_i(\rho_i))}{g_i'(\ell(\rho_i))}. \tag{8}$$

Because the productiono function is homogeneous degree one, the profit maximizing production level is determined only by the proportion, given the price vector $(\rho_i, p_i(\rho_i))$, and the level can be determined by the demand side, i.e., it is determined at the level that satisfies the condition of equilibrium of commodity market i,

$$K_i(\rho) = \alpha_i \frac{\rho_1 K_1 + \rho_2 K_2 + L}{p_i(\rho_i) F_i(1, \ell_i(\rho_i))} = \alpha_i \frac{g_i'(\ell_i(\rho_i))}{g_i(\ell_i(\rho_i))} (\rho_1 K_1 + \rho_2 K_2 + L) \tag{9}$$

and it determines the demand for the capital $K_i(\rho), i = 1, 2$. The demand for labor is $L_i(\rho) = \ell_i(\rho_i) K_i(\rho)$, and the pair $(K_i(\rho), L_i(\rho))$ satisfies the marginal condition for the optimality , $i = 1, 2$.

$$K_i(\rho),\ L_i(\rho),\ Y_i(\rho) = F_i(K_i(\rho), L_i(\rho)), i = 1, 2 \tag{10}$$

are the factor demand function and the commodity supply function. From these definition, We can see that the equilibrium condition of the commodity market

$$p_i(\rho_i) F_i(K_i(\rho), L_i(\rho)) = \alpha_i(\rho_1 K_1 + \rho_2 K_2 + L),\ i = 1, 2 \tag{11}$$

is an identity with respect to ρ. From the equilibrium condition in the factor market

$$K_i(\rho) = K_i,\ i = 1, 2, \quad L_1(\rho) + L_2(\rho) = L. \tag{12}$$

Let us confirm that Walris Law holds.

$$\begin{aligned}
&\rho_1(K_1(\rho) - K_1) + \rho_2(K_2(\rho) - K_2) + (L_1(\rho) + L_2(\rho) - L) \\
&= (\rho_1 K_1(\rho) + L_1(\rho)) + (\rho_2 K_2(\rho) + L_2(\rho)) - (\alpha_1 + \alpha_2)(\rho_1 K_1 + \rho_2 K_2 + L) \\
&= \{p_1(\rho) F_1(K_1(\rho), L_1(\rho)) - \alpha_1(\rho_1 K_1 + \rho_2 K_2 + L)\} \\
&\quad + \{p_2(\rho) F_2(K_2(\rho), L_1(\rho)) - \alpha_2(\rho_1 K_1 + \rho_2 K_2 + L)\} = 0
\end{aligned}$$

The equilibrium of $\mathcal{E}_2$ can be obtained by solving (12).

2.4 Existence of an Equilibrium

From the discussion in the previous subsection, it can be said that the general equilibrium in production exists if there exists $\rho^* = (\rho_1^*, \rho_2^*)$ that achieves the equilibrium in the sector specific capital markets, i.e. $K_i(\rho) = K_i,\ i = 1, 2$. From the capital markets' equilibrium conditions of (4), (9), we obtain

$$\begin{aligned}
\rho_1 K_1 + \rho_2 K_2 &= \rho_1 K_1(\rho) + \rho_2 K_2(\rho) \\
&= \left(\alpha_1 \rho_1 \frac{g_1'}{g_1} + \alpha_1 \rho_1 \frac{g_2'}{g_2}\right)(\rho_1 K_1 + \rho_2 K_2 + L) \\
&= \left(\alpha_1 \frac{g_1 - \ell_1 g_1'}{g_1} + \alpha_2 \frac{g_2 - \ell_2 g_2'}{g_2}\right)(\rho_1 K_1 + \rho_2 K_2 + L).
\end{aligned}$$

Here, for the sake of simplicity, we abuse the notation as g_i, g_i', ℓ_i instead of $g_i(\ell_i(\rho_i)), g_i'(\ell_i(\rho_i)), \ell_i(\rho_i)$.

From the first term of the RHS, if we use $\phi(\rho) = \alpha_1 \frac{g_1 - \ell_1 g_1'}{g_1} + \alpha_2 \frac{g_2 - \ell_2 g_2'}{g_2}$, the balance between the demand and supply in the capital market i becomes

$$\begin{aligned}
\phi(\rho) &= \alpha_1\left(1 - \frac{\ell_1}{\rho_1 + \ell_1}\right) + \alpha_2\left(1 - \frac{\ell_2}{\rho_2 + \ell_2}\right) \\
&= 1 - \alpha_1 \frac{\ell_1}{\rho_1 + \ell_1} - \alpha_2 \frac{\ell_2}{\rho_2 + \ell_2} \\
1 - \phi(\rho) &= \alpha_1 \frac{\ell_1}{\rho_1 + \ell_1} + \alpha_2 \frac{\ell_2}{\rho_2 + \ell_2} = \sum_{i=1}^{2} \alpha_i \frac{\ell_i}{\rho_i + \ell_i}.
\end{aligned}$$

Therefore, we have

$$\rho_1 K_1 + \rho_2 K_2 + L = \frac{1}{1 - \phi(\rho)} L.$$

From, (9), (12), the demand-supply balance in the capital market becomes

$$\alpha_i \frac{g_i'}{g_i} \frac{1}{1-\phi(\rho)} L = K_i$$

From (5)

$$\alpha_i \frac{1}{\rho_i + \ell_i} \frac{1}{1-\phi(\rho)} L = K_i$$

Substitute the definition of ϕ, and we have

$$\alpha_i \frac{1}{\rho_i + \ell_i} \frac{1}{\alpha_1 \frac{\ell_1}{\rho_1 + \ell_1} + \alpha_2 \frac{\ell_2}{\rho_2 + \ell_2}} L = K_i.$$

By simple manipulations, we have

$$\frac{\ell_1(\rho_1)}{\rho_1 + \ell_1(\rho_1)} + \frac{\alpha_2}{\alpha_1} \frac{\ell_2(\rho_2)}{\rho_2 + \ell_2(\rho_2)} = \frac{L}{K_1} \frac{1}{\rho_1 + \ell_1(\rho_1)}, \tag{13}$$

$$\tag{14}$$

which can be applied to $i = 2$, so

$$\frac{\ell_2(\rho_2)}{\rho_2 + \ell_2(\rho_2)} + \frac{\alpha_1}{\alpha_2} \frac{\ell_1(\rho_1)}{\rho_1 + \ell_1(\rho_1)} = \frac{L}{K_2} \frac{1}{\rho_2 + \ell_2(\rho_2)}, \tag{15}$$

and

$$\ell_2(\rho_2) + \frac{\alpha_1}{\alpha_2} \frac{\ell_1(\rho_1)(\rho_2 + \ell_2(\rho_1))}{\rho_1 + \ell_1(\rho_1)} = \frac{L}{K_2}$$

From (13) and (15) we obtain Therefore,

$$\frac{\alpha_1}{\alpha_2} \frac{K_2}{K_1} \frac{1}{\rho_1 + \ell_1(\rho_1)} = \frac{1}{\rho_2 + \ell_2(\rho_2)}$$

Putting this back to (13), and we have

$$\frac{\ell_1(\rho_1)}{\rho_1 + \ell_1(\rho_1)} + \frac{\alpha_2}{\alpha_1} \frac{\alpha_1}{\alpha_2} \frac{K_2}{K_1} \frac{\ell_2(\rho_2)}{\rho_1 + \ell_1(\rho_1)} = \frac{L}{K_1} \frac{1}{\rho_1 + \ell_1(\rho_1)}$$
$$\ell_1(\rho_1) + \frac{K_2}{K_1} \ell_2(\rho_2) = \frac{L}{K_1}.$$

By arranging the last equation, we can separate ρ_1 and ρ_2 and obtain

$$\ell_1(\rho_1) = \frac{L}{K_1} - \frac{K_2}{K_1}\ell_2(\rho_2). \tag{16}$$

Now we are showing that (ρ_1, ρ_2) is determined to satisfy (16). $\ell_i(\rho_i)$ is a monotonically increasing function of ρ_i, and

$$\lim_{\rho_i \to 0} \ell_i(\rho_i) = 0, \ \lim_{\rho_i \to \infty} \ell_i(\rho_i) = \infty, \ i = 1, 2.$$

It means that

$$\exists \bar{\rho}_i > 0 : \quad L - K_i \ell_i(\bar{\rho}_i) = 0, i = 1, 2,$$

and for $\rho_2 \in (0, \bar{\rho}_2)$, we define a inverse function of

$$\psi(\rho_2) = {\ell_1}^{-1}(L/K_1 - \ell_2(\rho_2)K_2/K_1)$$

and the relationship (16) becomes into

$$\rho_1 = \psi(\rho_2).$$

ψ has the properties as

$$\lim_{\rho_2 \to 0} \psi(\rho_2) = \ell_1^{-1}(L/K_1) = \bar{\rho}_1, \ \lim_{\rho_2 \to \bar{\rho}_2} \psi(\rho_2) = 0,$$
$$\psi' = \left(\ell_1^{-1}\right)' \times \left(-\frac{K_2}{K_1}\ell_2'(\rho_2)\right) = -\frac{\ell_2'}{\ell_1'}\frac{K_2}{K_1} < 0.$$

or

$$\psi'(\rho_2) = -\frac{g_1 g_1''}{(g_1)^2}\frac{(g_2)^2}{g_2 g_2''}\frac{K_2}{K_1} < 0.$$

From (13)

$$\begin{aligned}
&\frac{\ell_2(\rho_2)}{\rho_2 + \ell_2(\rho_2)} + \frac{\alpha_1}{\alpha_2}\frac{\ell_1(\rho_1)}{\rho_1 + \ell_1(\rho_1)} = \frac{L}{K_2}\frac{1}{\rho_2 + \ell_2(\rho_2)} \\
&\frac{\ell_1(\rho_1)}{\rho_1 + \ell_1(\rho_1)} = \frac{\alpha_2}{\alpha_1}\frac{L/K_2 - \ell_2(\rho_2)}{\rho_2 + \ell_2(\rho_2)} = \frac{\alpha_2}{\alpha_1}\frac{K_1}{K_2}\frac{L/K_1 - \ell_2(\rho_2)K_2/K_1}{\rho_2 + \ell_2(\rho_2)} \\
&\rho_1 + \ell_1(\rho_1) = \frac{\alpha_2}{\alpha_1}\frac{K_2}{K_1}(\rho_2 + \ell_2(\rho_2)) \\
&\psi(\rho_2) + \ell_1(\psi(\rho_2)) = \frac{\alpha_1}{\alpha_2}\frac{K_2}{K_1}\frac{\rho_2 + \ell_2(\rho_2)}{K_2}.
\end{aligned} \tag{17}$$

(17) is a equality with respect to ρ_2. If $\rho_2 = 0$,

$$\begin{aligned} \text{LHS of (17)} &= \psi(0) + \ell_1^{-1}(\psi(0))) = \ell^{-1}(L/K_1) > 0 \\ \text{RHS of (17)} &= 0, \end{aligned}$$

and we can see that RHS > LHS. If $\rho_2 = \bar{\rho}_2$,

$$\begin{aligned} \text{LHS of (17)} &= \psi(\bar{\rho}_2) + \ell_1^{-1}(\psi(\bar{\rho}_2))) = 0 \\ \text{RHS of (17)} &= \frac{\alpha_1}{\alpha_2}\frac{K_2}{K_1}(\bar{\rho}_2 + L/K_2) > 0, \end{aligned}$$

and we can see that RHS<LHS. By wrapping up the results of the discussion above, we can say that LHS is strictly decreasing and RHS is strictly increasing. Therefore, we successfully showed that a unique $\rho_2^* \in (0, \bar{\rho}_2)$ that satisfies (17) exists. Let $\rho_1^* = \psi(\rho_2^*)$ and we can say that (ρ_1^*, ρ_2^*) balances the capital markets.

Theorem 1 *There exists a unique general equilibrium of a two-sector economy with heterogeneous capitals and the production functions that satisfy* Assumption 1.。

From the discussion above, we can describe the solution of (12) as

$$\rho_i(K_1, K_2, L),\ i = 1, 2.$$

2.5 Exixtence of an Equilibrium – Another Approach

In this section we prove the existence of an equilibrium in $\mathcal{E}_2$-economy in another fashion. Although showing this proof may look redundant, we still show this work because following Theorem 2 is substantially interesting.
Let us define an artificial maximization problem such as follows:

$$\max_{(y_i, L_i)_{i=1}^2} \left(\frac{y_1}{\alpha_1}\right)^{\alpha_1} \left(\frac{y_2}{\alpha_2}\right)^{\alpha_2} \text{ subject to } \begin{cases} L_1 + L_2 \leqq L, \\ y_1 \leqq F_1(K_1, L_1), \\ y_2 \leqq F_2(K_2, L_2). \end{cases} \tag{18}$$

Note that L_1, L_2, y_1, y_2 are the endogenous variables of the artificial maximization problem (18) and K_1, K_2 are the exogenous ones because they are fixed by the amounts of initial endowment. Let A be a set of all $(y_i, L_i)_{i=1}^2$

satisfying constraints in (18). The set A represents the attainable set of the economy and is convex and compact. We can observe that an upper contour set of the objective function in (18) is a strictly convex set in $\mathbb{R}^2_{++}$. The production pair (y_1^*, y_2^*) of the solution is unique. Capitals K_1 and K_2 are constants. And thus, the solution to the problem (18) exists uniquely. Therefore next Lemma is obvious.

Lemma 1 *A solution to the problem* (18) *exists uniquely and is positive.*

The Lagrangian of the problem (18) is

$$\mathcal{L} = \left(\frac{y_1}{\alpha_1}\right)^{\alpha_1} \left(\frac{y_2}{\alpha_2}\right)^{\alpha_2} + \sum_{i=1}^{2} \lambda_i (F_i(K_i, L_i) - y_i) + \delta(L - L_1 - L_2).$$

The Kuhn-Tucker condition associating with (18) is a following set of equations:

$$\alpha_i \frac{(y_1/\alpha_1)^{\alpha_1}(y_2/\alpha_2)^{\alpha_2}}{y_i} - \lambda_i = 0, \; i = 1, 2 \tag{19}$$

$$\lambda_i \frac{\partial F_i}{\partial L_i} - \delta = 0, \qquad i = 1, 2 \tag{20}$$

$$F_i(K_i, L_i) - y_i = 0, \qquad i = 1, 2 \tag{21}$$

$$L - L_1 - L_2 = 0. \tag{22}$$

Let $(y_i^{**}, L_i^{**})_{i=1}^2$ be the solution to (18). Substituting (y_1^{**}, y_2^{**}) to (y_1, y_2) in (19), we have values of $\lambda_i, i = 1, 2$, and sequentially, we have δ from (20). Clearly, the allocation $(y_i^{**}, L_i^{**})_{i=1}^2$ satisfies (21) and (22).

Theorem 2 (Equivalence Theorem) *This equilibrium is equivalent to the solution to* (18)*. That is to say, following conditions* (i) *and* (ii) *hold.*

(i) *Let* $((\rho_i^*, p_i^*)_{i=1}^2, (Y_i^*, K_i^*, L_i^*)_{i=1}^2)$ *be the general equilibrium of* $\mathcal{E}_2$*-economy. Define*

$$p^* = p_1^{*\alpha_1} p_2^{*\alpha_2} \tag{23}$$

$$\lambda_i = \frac{p_i^*}{p^*}, i = 1, 2, \; \delta = \frac{1}{p^*}. \tag{24}$$

Then the pair $((\lambda_1, \lambda_2, \delta), (Y_i^*, L_i^*)_{i=1}^2)$ *is a solution to the system* (19), (20), (21) *and* (22)*. Furthermore,* $(Y_i^*, L_i^*)_{i=1}^2$ *is a solution to the problem* (18)*.*

(ii) *Let* $((\lambda_1, \lambda_2, \delta), (y_i^{**}, L_i^{**})_{i=1}^2)$ *be a solution of Kuhn-Tucker condition associating with* (18). *Define*

$$\rho_i^* = \frac{\lambda_i^*}{\delta} \frac{\partial F_i}{\partial \tilde{K}_i}(K_i, L_i^{**}), \; p_i^* = \frac{\lambda_i^*}{\delta}, \; i = 1, 2, \tag{25}$$

then $\delta^{-1} = p_1^{*\alpha_1} p_2^{*\alpha_2}$ *and the pair* $((\rho_i^*, p_i^*)_{i=1}^2), (y_i^{**}, K_i, L_i^{**})_{i=1}^2)$ *is an equilibrium.*

[Proof]

Proof of (i). Let $((\rho_i^*, p_i^*)_{i=1}^2), (Y_i^*, K_i^*, L_i^*)_{i=1}^2)$ be an equilibrium. By (24), the second equality in Kuhn-Tucker condition (20) hold:

$$\lambda_i \frac{\partial F_i}{\partial L_i}(K_i^*, L_i^*) - \delta = \frac{1}{p^*}\left(p_i^* \frac{\partial F_i}{\partial L_i}(K_i^*, L_i^*) - 1\right) = 0.$$

Equalities (21) and (22) hold obviously. The equilibrium of commodity i enables us to know:

$$\begin{aligned} \alpha_i \left(\frac{Y_1^*}{\alpha_1}\right)^{\alpha_1} \left(\frac{Y_2^*}{\alpha_2}\right)^{\alpha_2} - \lambda_i Y_i^* &= \frac{1}{p^*}\left(\alpha_i \left(\frac{p_1^* Y_1^*}{\alpha_1}\right)^{\alpha_1} \left(\frac{p_2^* Y_2^*}{\alpha_2}\right)^{\alpha_2} - p_i^* Y_i^*\right) \\ &= \frac{1}{p^*}\left(\alpha_i(\rho_1^* K_1 + \rho_2^* K_2 + L) - p_i^* Y_i^*\right) = 0. \end{aligned}$$

Therefore we have

$$\alpha_i \frac{\left(\dfrac{Y_1^*}{\alpha_1}\right)^{\alpha_1 - 1} \left(\dfrac{Y_2^*}{\alpha_2}\right)^{\alpha_2}}{Y_i^*} - \lambda_i = 0, \; i = 1, 2.$$

This implies that the first assertion in (i) holds. Finally, $(Y_i^*, L_i^*)_{i=1}^2$ satisfying the Kuhn-Tucker condition (19), (20), (21), and (22) is a solution to the problem (18) by Magasarian (1969, Theorem 7.2.1). This is the second assertion in (i).

Proof of (ii). Let $((\lambda_1, \lambda_2, \delta), (y_i^{**}, L_i^{**})_{i=1}^2)$ be a solution to Kuhn-Tucker condition associating to (18). Note that these variables are positive. By (25), we have $(\rho_i^*, \; p_i^*), i = 1, 2$ satisfying

$$p_i^* \frac{\partial F_i}{\partial K_i}(K_i, L_i^{**}) = \rho_i^*, \quad i = 1, 2.$$

Equalities (20) are rearranged to

$$p_i^* \frac{\partial F_i}{\partial L_i}(K_i, L_i^{**}) = 1.$$

These two equalities imply that (K_i, L_i^{**}) is a profit maximizer of the producer i when prices are (ρ_i^*, p_i^*). Labor market is in balance because (22) holds. Let us consider the commodity markets. By (19), we have

$$\left(\frac{y_1^{**}}{\alpha_1}\right)^{\alpha_1} \left(\frac{y_2^{**}}{\alpha_2}\right)^{\alpha_2} = \lambda_i \frac{y_i^{**}}{\alpha_i},\ i = 1, 2.$$

This equality implies $1 = \prod_{i=1}^{2} \lambda_i^{\alpha_i}$. By the definitions of ρ_i^* and $p_i^*, i = 1, 2$ in (25) we can obtain $\delta^{-1} = \prod_{i=1}^{2} (p_i^*)^{\alpha_i}$. Furthermore, by (21) we have:

$$y_i^{**} = F_i(K_i, L_i^{**}) = \frac{\partial F_i}{\partial \tilde{K}_i}(K_i, L_i^{**})K_i + \frac{\partial F_i}{\partial L_i}(K_i, L_i^{**})L_i^{**},$$
$$p_i^* y_i^{**} = \rho_i^* K_i + L_i^{**}.$$

Multiplying both sides of (19) by $\prod_{i=1}^{2} (p_i^*)^{\alpha_i}$ $(= \delta^{-1})$, we have

$$\alpha_i \left(\frac{p_1^* y_1^{**}}{\alpha_1}\right)^{\alpha_1} \left(\frac{p_2^* y_2^{**}}{\alpha_2}\right)^{\alpha_2} = p_i^* y_i^{**},\ i = 1, 2. \tag{26}$$

Add up above equalities with respect to i and we obtain

$$\left(\frac{p_1^* y_1^{**}}{\alpha_1}\right)^{\alpha_1} \left(\frac{p_2^* y_2^{**}}{\alpha_2}\right)^{\alpha_2} = p_1^* y_1^{**} + p_2^* y_2^{**} = \rho_1^* K_1 + \rho_2^* K_2 + L.$$

We can substitute this into the same term in (26). And taking (21) into account, we finally arrive at a conclusion:

$$\alpha_i \left(\rho_1^* K_1 + \rho_2^* K_2 + L\right) = p_i^* y_i^{**} = p_i^* F_i(K_i^{**} L_i^{**}),\ i = 1, 2.$$

This implies that two commodity markets are in balance. And thus, a pair of prices and allocation $\left((\rho_i^*, p_i^*)_{i=1}^2 , (y_i^{**}, L_i^{**})_{i=1}^2 \right)$ is the equilibrium of $\mathcal{E}_2$-economy. ∎

The unique solution of this artificial problem (18), $(y_i^{**}, L_i^{**})_{i=1}^2)$ is dependent on given parameters (K_1, K_2, L), and we describe this dependence by the function

$$y_i[K_1, K_2, L], L_i[K_1, K_2, L], i = 1, 2$$

3 Aggregated Economy

We suppose the economy that is prepared in the previous section $\mathcal{E}_2 = ((K_1, K_2, L),\ (F_i)_{i=1}^2,\ (\alpha_i)_{i=1}^2)$. The target of this section is to aggregate $\mathcal{E}_2$-economy to the one-sector economy $\mathcal{E}_1 = ((K_1, K_2, L), F)$. We assume that the initial endowments $(K_1, K_2, L) \in \mathbb{R}^3_{++}$ of $\mathcal{E}_1$ and $\mathcal{E}_2$ are identical. Let F be a production function with 3 variables, denoted as $F(\tilde{K}_1, \tilde{K}_2, \tilde{L})$. We assume that $F(\tilde{K}_1, \tilde{K}_2, \tilde{L})$ is continuous and homogeneous of degree one over $\mathbb{R}^3_+$, and that it is strictly increase and twice differentiable over $\mathbb{R}^3_{++}$.

Definition 2 *Consider an economy $\mathcal{E}_1 = ((K_1, K_2, L), F)$ where $F(\tilde{K}_1, \tilde{K}_2, \tilde{L})$. A pair of price and allocation $((\rho_1^{**}, \rho_2^{**}, p^{**}), (Y^{**}, K_1, K_2, L))$ is said to be an equilibrium of $\mathcal{E}_1$ if it satisfies the following conditions,*

$$
\begin{aligned}
& p^{**}\frac{\partial F}{\partial \tilde{K}_i}(K_1, K_2, L) = \rho_i^{**},\ \ i = 1, 2, \\
& p^{**}\frac{\partial F}{\partial \tilde{L}}(K_1, K_2, L) = 1, \\
& p^{**}Y^{**} = \rho_1^{**}K_1 + \rho_2^{**}K_2 + L, \\
& Y^{**} = F(K_1, K_2, L).
\end{aligned}
$$

The first two conditions of definition 2 mean that (K_1, K_2, L) is the quantity of factor demanded given $(\rho_1^*, \rho_2^*, p^{**})$. The latter two conditions mean that the demand and supply of the commodity market are in balance. Therefore, if the production function F has continuity, homogeneity of degree one, and differentiability, the existence of an equilibrium is preserved by definition,

Let the factor demand function that we obtain from (10) be $K_i(\rho), i = 1, 2,\ L(\rho), \rho = (\rho_1, \rho_2)$.

3.1 Definition of Aggregation

Definition 3 (Aggregation consistency 1) *Given the* **price of aggregated commodity** *$p(\rho)$ as a function of $\rho = (\rho_1, \rho_2) \in \mathbb{R}^2_{++}$, if there exists a production function $F(\tilde{K}_1, \tilde{K}_2, \tilde{L})$ that satisfies the following condition* (D)*, $\mathcal{E}_1$-economy is said to be* **consistent** *with $\mathcal{E}_2$-economy, $p(\rho)$ is said to be* **price of the aggregated commodity***, and $F(\tilde{K}_1, \tilde{K}_2, \tilde{L})$ is said to be the* **sector aggregated function***.*

(D) *for any* $\rho > (0,0)$, $K_1(\rho), K_2(\rho), L(\rho)$ *is a solution of the following problem of*

$$\max_{\tilde{K}_1, \tilde{K}_2, \tilde{L}} p(\rho)F(\tilde{K}_1, \tilde{K}_2, \tilde{L}) - \rho_1 \tilde{K}_1 - \rho_2 \tilde{K}_2 - \tilde{L}.$$

Condition (D) requires that the sum of the factor demands equals the solution of the aggregate producer's profit maximization problem for any price. We call the pair $(p(\rho), F(\tilde{K}_1, \tilde{K}_2, L))$ as **aggregate pair**, and the aggregate pair that preserves the consistency between the two economy as **consistent aggregate pair** . Moreover, we say that $\mathcal{E}_1$ is **locally consistent** with $\mathcal{E}_2$ for $\bar{\rho}$, if (D) is satisfied for a specific $\bar{\rho}$.

Definition 4 *(Aggregation to a one-capital and two-sector economy)* $\mathcal{E}_2$*Let the general equilibrium of the economy is* $((\rho_i^*, p_i^*)_{i=1}^2, (Y_i^*, K_i, L_i^*)_{i=1}^2)$. $\mathcal{E}_2$*-economy is said to be aggregated to* $\mathcal{E}_1$*-economy if and only if the following conditions* (C), (E) *are satisfied;*

(C) $\mathcal{E}_1$*-economy is consistent with* $\mathcal{E}_2$*-economy, and*

(E) *Let the consistent aggregate pair* を $(p(\rho), F(\tilde{K}_1, \tilde{K}_2, \tilde{L}))$ *that satisfies condition (C), The pair of the price and the allocation* $((\rho_1^*, \rho_2^*, p(\rho^*)),$ $((p_1^* Y_1^* + p_2^* Y_2^*)/p(\rho^*), K_1, K_2, L))$ *is the general equilibrium* $\mathcal{E}_1$*-economy.*

3.2 Basic Results

The definition of aggregation of definition 4 have the following properties.

Lemma 2 *Given* $\mathcal{E}_2$*-economy. If* $\mathcal{E}_1$*-economy is consistent with* $\mathcal{E}_2$*-economy given an aggregate pair* $(p(\rho), F(\tilde{K}_1, \tilde{K}_2, \tilde{L}))$*, then for any* ρ

$$\begin{aligned} & p(\rho)F(K_1(\rho), K_2(\rho), L(\rho)) \\ & \quad = p_1(\rho_1)F_1(K_1(\rho), L_1(\rho)) + p_2(\rho_2)F_2(K_2(\rho), L_2(\rho)) \\ & \quad = p_1(\rho_1)^{\alpha_1} p_2(\rho_2)^{\alpha_2} \left(\frac{F_1(K_1(\rho), L_1(\rho))}{\alpha_1} \right)^{\alpha_1} \left(\frac{F_2(K_2(\rho), L_2(\rho))}{\alpha_2} \right)^{\alpha_2} \end{aligned} \tag{27}$$

holds.

[Proof] For any $\rho = (\rho_1, \rho_2) \in \mathbb{R}^2_{++}$, from the homogeneity of degree one of F, F_1, F_2 and the fist order conditions of profit maximization, we have

$$
\begin{aligned}
&p(\rho)F(K_1(\rho), K_2(\rho), L(\rho)) \\
&\quad = p(\rho)\frac{\partial F}{\partial \tilde{K}_1}K_1(\rho) + p(\rho)\frac{\partial F}{\partial \tilde{K}_2}K_2(\rho) + p(\rho)\frac{\partial F}{\partial \tilde{L}}L(\rho) \\
&\quad = \rho_1 K_1(\rho) + \rho_2 K_2(\rho) + L(\rho) \\
&\quad = \{\rho_1 K_1(\rho) + L_1(\rho)\} + \{\rho_2 K_2(\rho) + L_2(\rho)\} \\
&\quad = p_1(\rho_1)F_1(K_1(\rho), L_1(\rho)) + p_2(\rho_2)F_2(K_2(\rho), L_2(\rho)).
\end{aligned}
$$

This is the first equation of Eq, (27). It also holds from (11)that

$$
\begin{aligned}
&p_1(\rho)F_1(K_1(\rho), L_1(\rho)) + p_2(\rho)F_2(K_2(\rho), L_2(\rho)) \\
&\quad = \alpha_1(\rho_1 K_1 + \rho_2 K_2 + L) + \alpha_2(\rho_1 K_1 + \rho_2 K_2 + L) \\
&\quad = (\rho_1 K_1 + \rho_2 K_2 + L)^{\alpha_1+\alpha_2} \\
&\quad = \left(\frac{\alpha_1(\rho_1 K_1 + \rho_2 K_2 + L)}{\alpha_1}\right)^{\alpha_1} \left(\frac{\alpha_2(\rho_1 K_1 + \rho_2 K_2 + L)}{\alpha_2}\right)^{\alpha_2} \\
&\quad = p_1(\rho_1)^{\alpha_1} p_2(\rho_2)^{\alpha_2} \left(\frac{F_1(K_1(\rho), L_1(\rho))}{\alpha_1}\right)^{\alpha_1} \left(\frac{F_2(K_2(\rho), L_2(\rho))}{\alpha_2}\right)^{\alpha_2}.
\end{aligned}
$$

This is the second equation of (27). ∎

This Lemma 2 drugs our attention from the perspective that the sum of the monetary values of the commodities $p_1(\rho_1)F_1(K_1(\rho), L_1(\rho)) + p_2(\rho_2)F_2(K_2(\rho), L_2(\rho))$ equals to the total monetary value of the aggregated commodity $p(\rho)F(K_1(\rho), K_2(\rho), L(\rho))$ ex-ante, if the consistent aggregation satisfying the condition (D) is possible.

Theorem 3 *(Consistency⇒Aggregation Possibility) Let $\mathcal{E}_2$-economy be given. If $\mathcal{E}_1$-economy is consistent with $\mathcal{E}_2$-economy, then $\mathcal{E}_2$-economy is aggregated to $\mathcal{E}_1$-economy.*

[Proof] Given the consistent aggregate pair as $(p(\rho), F(\tilde{K}_1, \tilde{K}_2, \tilde{L}))$. $p(\rho)$ is a function of $\rho = (\rho_1, \rho_2)$, suppose that the general equilibrium of the economy $((K_1, K_2, L), (F_i)_{i=1}^2, (\alpha_i)_{i=1}^2)$ is $((p_i^*, \rho_i^*)_{i=1}^2, (Y_i^*, K_i, L_i^*)_{i=1}^2)$. Then, from lemma 2, we have

$$
\begin{aligned}
&p_1^* F_1(K_1, L_2^*) + p_2^* F_2(K_2, L_2^*) \\
&\quad = \rho_1^* K_1 + \rho_2^* K_2 + L = p(\rho^*)F(K_1(\rho^*), K_2(\rho^*), L(\rho^*)).
\end{aligned}
$$

Because $Y_i^* = F_i(K_i, L_i^*),\ i = 1, 2$, we can have the demand and supply of the aggregate commodity balanced, Therefore, by letting $Y^* = F(K_1(\rho^*), K_2(\rho^*), L(\rho^*))$, we have $Y^* = (p_1^* Y_1^* + p_2^* Y_2^*)/p(\rho^*)$, and the general equilibrium of $\mathcal{E}_1$-economy is $((p(\rho^*), \rho^*), (Y^*, K_1, K_2, L))$. ∎

We utilize the result of lemma 2 as

$$\begin{aligned} & p(\rho)F(K_1(\rho), K_2(\rho), L(\rho)) \\ & = p_1(\rho_1)^{\alpha_1} p_2(\rho_2)^{\alpha_2} \left(\frac{F_1(K_1(\rho), L_1(\rho))}{\alpha_1}\right)^{\alpha_1} \left(\frac{F_2(K_2(\rho), L_2(\rho))}{\alpha_2}\right)^{\alpha_2}. \end{aligned} \tag{28}$$

This equation implies that we have candidates of the price and the function of aggregate commodity given by

$$p(\rho) = p_1(\rho_1)^{\alpha_1} p_2(\rho_2)^{\alpha_2},\ F(K_1, K_2, L) = \left(\frac{F_1(K_1, L_1^*)}{\alpha_1}\right)^{\alpha_1} \left(\frac{F_2(K_2, L_2^*)}{\alpha_2}\right)^{\alpha_2}. \tag{29}$$

We obtain the candidate production function $F(K_1, K_2, L)$ of aggregate commodity by substituting the prices in equilibrium $\rho^* = (\rho_1^*, \rho_2^*)$ to (28). Because $\rho_i^*,\ i = 1, 2$ is a function of the initial endowments (K_1, K_2, L), we can say that F is a composite function of (K_1, K_2, L). By using $\rho_i^* = \rho_i(K_1, K_2, L), i = 1, 2$, we can describe F expilicitly as

$$F(K_1, K_2, L) = \left(\frac{F_1(K_1, L_1(\rho^*))}{\alpha_1}\right)^{\alpha_1} \left(\frac{F_2(K_2, L_2(\rho^*)}{\alpha_2}\right)^{\alpha_2} \tag{30}$$

$$L_i(\rho^*) = \ell_i(\rho_i^*)K_i,\ i = 1, 2,\ L_1(\rho^*) + L_2(\rho^*) = L \tag{31}$$

$$K_i = K_i(\rho^*),\ i = 1, 2. \tag{32}$$

Let us explain the procedure of "seeing $F(K_1, K_2, L)$ as a function of (K_1, K_2, L).

(s1) Consider a given initial endowment $\tilde{X} = (\tilde{K}_1, \tilde{K}_2, \tilde{L}) \in \mathbb{R}^3_{++}$.

(s2) Consider an $\mathcal{E}_2(\tilde{X}) = (\tilde{X}, (F_i)_{i=1}^2)$ that shares the production function $(F_i)_{i=1}^2$ with the original two-sector economy $\mathcal{E}_2$ but different in the amount of the initial endowments.

(s3) Then, we can say that $\mathcal{E}_2(\tilde{X}) = (\tilde{X}, (F_i)_{i=1}^2)$ has an equilibrium, and that there is $F(\tilde{X})$ that corresponds to $\tilde{X}$ from (30).

(s4) (s1), (s2), and (s4) let us define a one-sector production function $F(\tilde{K}_1, \tilde{K}_2, \tilde{L})$ as is in (30).

At this stage the candidate function $F(K_1, K_2, L)$'s homogeneity of degree one in (K_1, K_2, L) is obvious but its concavity is not.

3.3 Concavity and Differentiability of F

Theorem 4 *The function $F(K_1, K_2, L)$ defined by* (30) *is increasing, homogeneous of degree one, and concave.*

[Proof] It is obvious that $F(K_1, K_2, L)$ is increasing in K_1, K_2 and L and homogeneous of degree one in (K_1, K_2, L).

First, we make sure of a following basic result. Let us define a function $h(z_1, z_2) = {z_1}^{\alpha_1}{z_2}^{\alpha_2}$, $(z_1, z_2) \in \mathbb{R}^2_+$. Clearly, h is concave. This implies that for any $z = (z_1, z_2)$, $z' = (z'_1, z'_2) \in \mathbb{R}^2_+$ the inequality $h(z/2 + z'/2) \geqq \frac{1}{2}h(z) + \frac{1}{2}h(z')$ holds. That is,

$$\prod_{i=1}^{2}\left(\frac{z_i}{2} + \frac{z'_i}{2}\right)^{\alpha_i} \geqq \frac{1}{2}\prod_{i=1}^{2}{z_i}^{\alpha_i} + \frac{1}{2}\prod_{i=1}^{2}{z'_i}^{\alpha_i}. \tag{33}$$

We pick two vectors $X = (K_1, K_2, L), X' = (K'_1 K'_2, L')$ arbitrarily in $\mathbb{R}^3_{++}$. For symbolic simplicity, we write

$$\begin{aligned}
&L^x_i = L_i[X],\ \ Y^x_i = F_i(K_i, L^x_i)\\
&L^{x\prime}_i = L_i[X^{x\prime}],\ Y^{x\prime}_i = F_i(K_i, L^{x\prime}_i),\ i = 1, 2.
\end{aligned}$$

It is obvious that $(K_1 + K'_1, K_2 + K'_2, \sum_{i=1}^{2}(L^x_i + L^{x\prime}_i)) = X + X'$. Then $(K_i + K'_i, L^x_i + L^{x\prime}_i)_{i=1}^2 \in A(X + X')$. By artificial maximization (18), we know that

$$\prod_{i=1}^{2}\left(\frac{F_i(K_i[X + X'], L_i[X + X'])}{\alpha_i}\right)^{\alpha_i} \geqq \prod_{i=1}^{2}\left(\frac{F_i(K_i + K'_i, L^x_i + L^{x\prime}_i)}{\alpha_i}\right)^{\alpha_i}.$$

Since $F_i(\cdot, \cdot)$ is homogeneous of degree one and concave, then next inequality holds.

$$\begin{aligned}
F_i(K_i + K'_i, L^x_i + L^{x\prime}_i) &= 2F_i(K_i/2 + K'_i/2, L^x_i/2 + L^{x}_i{}'/2)\\
&\geqq 2\left(\frac{1}{2}F_i(K_i, L^x_i) + \frac{1}{2}F_i(K'_i, L^{x\prime}_i)\right).
\end{aligned}$$

This together with (33) implies

$$\prod_{i=1}^{2}\left(\frac{F_i(K_i+K_i',L_i^x+L_i^{x\prime})}{\alpha_i}\right)^{\alpha_i} \geqq 2\prod_{i=1}^{2}\left(\frac{1}{2}\frac{F_i(K_i,L_i^x)}{\alpha_i}+\frac{1}{2}\frac{F_i(K_i',L_i^{x\prime})}{\alpha_i}\right)^{\alpha_i}$$
$$\geqq \prod_{i=1}^{2}\left(\frac{F_i\left(K_i,L_i^x\right)}{\alpha_i}\right)^{\alpha_i}+\prod_{i=1}^{2}\left(\frac{F_i(K_i',L_i^{x\prime})}{\alpha_i}\right)^{\alpha_i}.$$

The above inequality is written simply as

$$F(X+X') \geqq F(X)+F(X').$$

In addition to this F is homogeneous of degree one. Then F is a concave function. ∎

3.4 $\mathcal{E}_1$-economy's consitency to $\mathcal{E}_2$-economy

We assume that each function is differentiable. Note that production function F is a function of (K_1, K_2, L) but it takes a form of a triple composite function,

$$F(K_1,K_2,L)=\prod_{i=1}^{2}\left(\frac{F_i(K_i,L_i(\rho(K_1,K_2,L)))}{\alpha_i}\right)^{\alpha_i}.$$

Because, by applying the functional relationship that we found in section 2.5, we can see that $L_i(\rho(K_1,K_2,L)) = L_i[K_1,K_2,L]$ is a identity with respect to (K_1,K_2,L), we can reduce the triple composite function to the double composite function

$$F(K_1,K_2,L)=\prod_{i=1}^{2}\left(\frac{F_i(K_i,L_i[K_1,K_2,L])}{\alpha_i}\right)^{\alpha_i}.$$

Hereafter we utilize the above functional form because this makes our lives much easier.

3.4.1 Local Consistency

Now let us see the differentiation of the function F with respect to the amount of initial endowments (K_1, K_2, L).

$$\begin{aligned}
&\frac{\partial F}{\partial K_1}(K_1, K_2, L) \\
&= \left(\frac{\partial F_1}{\partial K_1} + \frac{\partial F_1}{\partial L_1}\frac{\partial L_1}{\partial K_1}\right)\alpha_1\left(\prod_{i=1}^{2}\left(\frac{F_i}{\alpha_i}\right)^{\alpha_i}\right)/F_1 + \left(\frac{\partial F_2}{\partial L_2}\frac{\partial L_2}{\partial K_1}\right)\alpha_2\left(\prod_{i=1}^{2}\left(\frac{F_i}{\alpha_i}\right)^{\alpha_i}\right)/F_2 \\
&= \left(\frac{\partial F_1}{\partial K_1} + \frac{\partial F_1}{\partial L_1}\frac{\partial L_1}{\partial K_1}\right)\lambda_1^* + \left(\frac{\partial F_2}{\partial L_2}\frac{\partial L_2}{\partial K_1}\right)\lambda_2^*
\end{aligned}$$

and

$$\begin{aligned}
p^*\frac{\partial F}{\partial K_1}(K_1, K_2, L) &= \delta^{*-1}\frac{\partial F}{\partial K_1}(K_1, K_2, L) \quad : \text{Note } p^* = p(\rho^*) \\
&= \left(\frac{\partial F_1}{\partial K_1} + \frac{\partial F_1}{\partial L_1}\frac{\partial L_1}{\partial K_1}\right)\frac{\lambda_1^*}{\delta^*} + \left(\frac{\partial F_2}{\partial L_2}\frac{\partial L_2}{\partial K_1}\right)\frac{\lambda_2^*}{\delta^*} \\
\text{From (25):} \qquad &= p_1^*\left(\frac{\partial F_1}{\partial K_1} + \frac{\partial F_1}{\partial L_1}\frac{\partial L_1}{\partial K_1}\right) + p_2^*\left(\frac{\partial F_2}{\partial L_2}\frac{\partial L_2}{\partial K_1}\right) \\
&= p_1^*\frac{\partial F_1}{\partial K_1} + p_1^*\frac{\partial F_1}{\partial L_1}\frac{\partial L_1}{\partial K_1} + p_2^*\frac{\partial F_2}{\partial L_2}\frac{\partial L_2}{\partial K_1} \\
&= \rho_1^* + \frac{\partial L_1}{\partial K_1} + \frac{\partial L_2}{\partial K_1} = \rho_1^*
\end{aligned}$$

Differentiation with respect to K_2 leads the same equation, and thus, we can say that

$$p^*\frac{\partial F}{\partial K_i}(K_1, K_2, L) = \rho_i^*, \ i = 1, 2 \tag{34}$$

This is the marginal condition of the profit maximization problem. Furthermore,

$$\begin{aligned}
&\frac{\partial F}{\partial L}(K_1, K_2, L) \\
&= \alpha_1\left(\frac{F_1}{\alpha_1}\right)^{\alpha_1}\left(\frac{F_2}{\alpha_2}\right)^{\alpha_2}\frac{1}{F_1}\frac{\partial F_1}{\partial L_1}\frac{\partial L_1}{\partial L} + \alpha_2\left(\frac{F_1}{\alpha_1}\right)^{\alpha_1}\left(\frac{F_2}{\alpha_2}\right)^{\alpha_2}\frac{1}{F_2}\frac{\partial F_2}{\partial L_2}\frac{\partial L_2}{\partial L} \\
&= \lambda_1\frac{\partial F_1}{\partial L_1}\frac{\partial L_1}{\partial L} + \lambda_2\frac{\partial F_2}{\partial L_2}\frac{\partial L_2}{\partial L}
\end{aligned}$$

and

$$\begin{aligned}
p^*\frac{\partial F}{\partial L}(K_1, K_2, L) = &\frac{1}{\delta^*}\frac{\partial F}{\partial L}(K_1, K_2, L) \\
=&\frac{\lambda_1}{\delta^*}\frac{\partial F_1}{\partial L_1}\frac{\partial L_1}{\partial L} + \frac{\lambda_2}{\delta^*}\frac{\partial F_2}{\partial L_2}\frac{\partial L_2}{\partial L} \\
=&p_1^*\frac{\partial F_1}{\partial L_1}\frac{\partial L_1}{\partial L} + p_2^*\frac{\partial F_2}{\partial L_2}\frac{\partial L_2}{\partial L} = 1,
\end{aligned}$$

and therefore, we obtain

$$p^*\frac{\partial F}{\partial L}(K_1, K_2, L) = 1. \tag{35}$$

From (34), and (35), the necessary condition for (K_1, K_2, L) to be the solution of the profit maximization problem of

$$\max p^* F(\tilde{K}_1, \tilde{K}_2, \tilde{L}) - \rho_1^*\tilde{K}_1 - \rho_2^*\tilde{K}_2 - \tilde{L}.$$

Because F is a concave function, (34), and (35) can be shown that they are the necessary and sufficient conditions of the maximization problem, if if the differentiability of F holds,

Now we can say that the next lemma holds.

Lemma 3 *(Local Consistency) Suppose the aggregate pair* $(p(\rho), F(\tilde{K}_1, \tilde{K}_2, \tilde{L}))$ *is a production function in a one-sector economy given as* (30). *Suppose that the one-sector economy has the initial endowments identical to the one of* $\mathcal{E}_2$, *and it is given as* $\mathcal{E}_1 = ((K_1, K_2, L), F)$, *its general equilibrium is* $((\rho_i^*)_{i=1}^2, p(\rho^*)), (K_1, K_2, L))$. *Namely,* $\mathcal{E}_1$ *is locally consistent with* $\mathcal{E}_2$ *at* ρ^*.

3.4.2 Trivial Equilibrium

Now, we introduce a concept of **trivial equilibrium**.

Theorem 5 *Let* $\rho = (\rho_1, \rho_2)$ *be a pair of arbitrary fixed positive rental-wage ratios. Then* $((\rho_i, p_i(\rho_i))_{i=1}^2, (Y_i(\rho), K_i(\rho), L_i(\rho))_{i=1}^2)$ *is a pair of price vectors and production vectors in the two sector economy* $\mathcal{E}_2 = ((K_1, K_2, L), (F_i)_{i=1}^2, (\alpha_i)_{i=1}^2)$. *Define* $\bar{X} = (\bar{K}_1, \bar{K}_2, \bar{L}) = (K_1(\rho), K_2(\rho), L_1(\rho) + L_2(\rho))$. *Then the pair*

$$\big((\rho_i, p_i(\rho))_{i=1}^2), (Y_i(\rho), K_i(\rho), L_i(\rho))_{i=1}^2\big)$$

is a equilibrium in a new economy $\mathcal{E}_2(\bar{X}) = (\bar{X}, (F_i)_{i=1}^2, (\alpha_i)_{i=1}^2)$. *This equilibrium is said to be a "trivial equilibrium". A trivial equilibrium is a solution to the problem* (18) *when a vector of initial holdings is* $\bar{X}$.

[Proof] Note that the labor capital ratios $\ell_i(\tilde{\rho}_i), i = 1, 2$ in $\mathcal{E}_2$ are identical with those in $\mathcal{E}_2(\bar{X})$, since F_i's are common in two economies. Second, let us consider the equilibrium of commodity markets. By (11) and the Euler's theorem on F_i in $\mathcal{E}_2$, it holds that

$$\rho_i K_i(\rho) + L_i(\rho) = p_i(\rho_i)g_i(\ell_i(\rho_i))K_i(\rho) = \alpha_i(\rho_1 K_1 + \rho_2 K_2 + L), i = 1, 2.$$

Adding these up with respect to i leads us to

$$\rho_1 \bar{K}_1 + \rho_2 \bar{K}_2 + \bar{L} = \rho_1 K_1(\rho) + \rho_2 K_2(\rho) + L(\rho) = \rho_1 K_1 + \rho_2 K_2 + L.$$

This fact leads us to

$$p_i(\rho_i)g_i(\ell_i(\rho_i))K_i(\rho) = \alpha_i(\rho_1 \bar{K}_1 + \rho_2 \bar{K}_2 + \bar{L}),\ i = 1, 2.$$

This implies that the demand for labor and capital of the i-th sector in $\mathcal{E}_2(\bar{X})$ are identical with $L_i(\rho)$ and $K_i(\rho)$ in $\mathcal{E}_2$, $i = 1, 2$. And thus in $\mathcal{E}_2(\bar{X})$ two factor markets are in balance at ρ. Then the pair of price and allocation $((\rho_i, p_i(\rho_i))_{i=1}^2, (Y_i\ (\rho), K_i(\rho), L_i(\rho))_{i=1}^2)$ is an equilibrium in $\mathcal{E}_2(\bar{X})$. The second assertion holds obviously. ∎

The next theorem is the major result of this study.

Theorem 6 (Possibility Theorem of Sector Aggregation) *With the aggregate pair* $(p(\rho) = p_1(\rho_1)^{\alpha_1} p_2(\rho_2)^{\alpha_2}$*,* $F(K_1, K_2, L))$ *given by* (29)*,* $\mathcal{E}_1$*-economy* $((K_1, K_2, L), F)$ *is consistent with* $\mathcal{E}_2$*-economy* $((K_1, K_2, L), (F_i)_{i=1}^2, (\alpha_i)_{i=1}^2)$*, Furthermore, the economy* $((K_1, K_2, L), (F_i)_{i=1}^2, (\alpha_i)_{i=1}^2)$ *is aggregated to the economy* $((K_1, K_2, L), F)$*.*

[Proof] Let the rental-wage ratio be $\rho = (\rho_1, \rho_2)$, and $\bar{X} = (K_1(\rho), K_2(\rho), L(\rho))$. From lemma 5, the pair of the price and the allocation$((\rho_i, p_i(\rho_i))_{i=1}^n, (Y_i(\rho), K_i(\rho), L_i(\rho))$ is the equilibrium of $\mathcal{E}_2(X)$. Therefore, from the lemma 3, $(K_1(\rho), K_2(\rho), L(\rho))$ is the solution of the problem of

$$\max_{\tilde{K}_i, \tilde{L}_i} p(\rho)F(\tilde{K}_i, \tilde{L}_i) - \rho_1 \tilde{K}_1 - \rho_2 \tilde{K}_2 - L \qquad ∎$$

4 Concluding Remarks

This study successfully shows the possibility of aggregating an economy that has two production sectors producing two heterogeneous (or sector specific) goods capitals with two heterogeneous (or sector specific) capitals to the economy that has one sector producing one good with one capital, which is so-called a macro economy. The highlight of this study lies in the aggregation of production function and the heterogeneous capitals with preserving the two types of consistency. The first consistency is the one between the factor demands in the economy before and after the aggregation. The second consistency is the total value of the goods in equilibrium of economies before and after the aggregation. We show that the aggregation of a economy is possible with satisfying these consistencies. The result of this study provides an affirmative answer to Cambridge controversy, which is reversal to the traditional answers such as Felipe and McCombie (2013).

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